Existence and Hypoellipticity for Partial Differential Operators on the Nilpotent Cartan-Lie Group G_5

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Abstract

The goal of this paper is to prove the existence theorem for any invariant differential operator on the nilpotent Lie group G_5. Out of the Hormander condition, we prove the hypoellipticity for a remarkable class of differential operators G_5.

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1 Introduction and results.

1.1. Let G_5 be the real group consisting of all matrices of the form

$$\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & -x_1 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & \frac{x_2^2}{2} & x_5 - \frac{x_1 x_2^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & x_2 & -x_3 - x_1x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, $x_3 \in \mathbb{R}$, $x_4 \in \mathbb{R}$ and $x_5 \in \mathbb{R}$. Let $K = \mathbb{R}^5$ be the group with the following law

$$\begin{aligned} & (x_5, x_4, x_3, x_2, x_1)(y_5, y_4, y_3, y_2, y_1) \\ = \; & (x_5 + y_5 + \frac{1}{2}x_1y_2^2 - x_2y_3 + x_1x_2y_2, x_4 + y_4 + \frac{1}{2}x_1^2y_2 - x_1y_3, y_3 + x_3 - x_1y_2, x_2 + y_2, x_1 + y_1) \end{aligned}$$

for any $(x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^5$ and $(y_5, y_4, y_3, y_2, y_1) \in \mathbb{R}^5$. The inverse of an element $(x_5, x_4, x_3, x_2, x_1)$ is

$$\begin{aligned} & (x_5, x_4, x_3, x_2, x_1)^{-1} \\ = \; & (-x_5 - \frac{x_1}{2}x_2^2 - x_2x_3, -x_4 - \frac{x_1^2}{2}x_2 - x_1x_3, -x_3 - x_1x_2, -x_2, -x_1) \end{aligned} \quad (2)$$

Dixmier had proved in $[8, P.331]$ that there is a group isomorphism between G_5 and K. Thanks to this isomorphism, the group K can be shown as a semidirect product $\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R} \underset{\rho_1}{\rtimes} \mathbb{R}$ of the real vector groups $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}^3$, where ρ_2 is the group homomorphism $\rho_2 : \mathbb{R} \to Aut(\mathbb{R}^3)$, which is defined by

$$\rho_2(x_2)(y_5, y_4, y_3) = (y_5 - x_2y_3, y_4, y_3) \quad (3)$$

and ρ_1 is the group homomorphism $\rho_1 : \mathbb{R} \to Aut(\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R})$, which is given by

$$\rho_1(x_1)(y_5, y_4, y_3, y_2) = (y_5 + \frac{x_1}{2}y_2^2, y_4 + \frac{x_1^2}{2}y_2 - x_1y_3, y_3 - x_1y_2, y_2) \quad (4)$$

where $Aut(\mathbb{R}^3)$ $(resp. Aut(\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R}))$ is the group of all automorphisms of $(\mathbb{R}^3)$ $(resp. (\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R}))$.

1.2. Let $C^\infty(K)$, $\mathcal{D}(K)$, $\mathcal{D}'(K)$, $\mathcal{E}'(K)$ be the space of C^∞- functions, C^∞ with compact support, distributions and distributions with compact support on G_5 respectively. If M is an unimodular Lie group, we denote by $L^1(M)$ the Banach algebra that consists of all complex valued functions on the group M, which are integrable with respect to the Haar measure of M and multiplication is defined by convolution on M , and we denote by $L^2(M)$ the Hilbert space of M. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of K; which is canonically isomorphic to the algebra of all distributions on K supported by $\{0\}$, where 0 is the identity element of K. For any $u \in \mathcal{U}$ one can define a differential operator P_u on K as follows:

$$P_u f(X) = u * f(X) = \int_K f(Y^{-1}X) u(Y) dY \qquad (5)$$

for any $f \in C^\infty(K)$, where $dY = dy_5 dy_4 dy_3 dy_2 dy_1$ is the Haar measure on K which is the Lebesgue measure on $\mathbb{R}^5$, $Y = (y_5, y_4, y_3, y_2, y_1)$, $X = (x_5, x_4, x_3, x_2, x_1)$ and $*$ denotes the convolution product on K.The mapping $u \rightarrow P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on K.

1.3. Let $B = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ be the group of the direct product of $\mathbb{R}^3, \mathbb{R}$ and $\mathbb{R}$. We denote also by $\mathcal{U}$ the complexified enveloping algebra of the real Lie algebra $\underline{b}$ of B. For every $u \in \mathcal{U}$, we can associate a differential operator Q_u on B as follows

$$\begin{aligned} Q_u f(X) &= u *_c f(X) = f *_c u(X) \\ &= \int_B f(X - Y) u(Y) dY \end{aligned} \qquad (6)$$

for any $f \in C^\infty(B)$, $X \in B, Y \in B$. where $*_c$ signify the convolution product on the real vector group B and $dY = dy_5 dy_4 dy_3 dy_2 dy_1$ is the Lebesgue measure on B. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on B, which are nothing but the algebra of differential operator with constant coefficients on B. For more details see[8, 18]

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2 Existence Theorem.

2.1. Let $L = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be the group with law:

$$\begin{aligned}
& X.Y \\
= \; & (x_5, x_4, x_3, x_2, t_2, x_1, t_1)(y_5, y_4, y_3, y_2, s_2, y_1, s_1) \\
= \; & (((x_5, x_4, x_3, x_2, t_2, x_1)(\rho_1(t_1)(y_5, y_4, y_3, s_2)), y_1 + x_1, t_1 + s_1) \\
= \; & ((x_5, x_4, x_3) + (\rho_2(t_2)(y_5 + \frac{t_1}{2}s_2^2, y_4 + \frac{t_1^2}{2}s_2 - t_1 y_3, y_3 - t_1 s_2)), \\
& x_2 + y_2, t_2 + s_2, y_1 + x_1, t_1 + s_1) \\
= \; & (x_5 + y_5 + \frac{t_1}{2}s_2^2 - t_2 y_3 + t_1 t_2 s_2, x_4 + y_4 + \frac{t_1^2}{2}s_2 - t_1 y_3, x_3 + y_3 - t_1 s_2, \\
& x_2 + y_2, t_2 + s_2, y_1 + x_1, t_1 + s_1) \qquad (7)
\end{aligned}$$

for all $X = (x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in L$ and $Y = (y_5, y_4, y_3, y_2, y_1, s_2, s_1) \in L$. In this case the group G_5 can be identified with the closed subgroup $\mathbb{R}^3 \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}$ of L and B with the subgroup $\mathbb{R}^3 \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}$ of L.

Definition 2.1. *For every $f \in C^\infty(G_5)$, one can define function $\widetilde{f} \in C^\infty(L)$ as follows:*

$$\begin{aligned}
& \widetilde{f}(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \qquad (8) \\
= \; & f((\rho_1(x_1)(\rho_2(x_2)(x_5, x_4, x_3))), x_2 + t_2), x_1 + t_1)
\end{aligned}$$

for all $(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in L$.

Remark 2.1. *The function $\widetilde{f}$ is invariant in the following sense:*

$$\begin{aligned}
& \widetilde{f}((\rho_1(h)((\rho_2(k)(x_5, x_4, x_3)), x_2 - k, t_2 + k)), x_1 - h, t_1 + h) \\
= \; & \widetilde{f}(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \qquad (9)
\end{aligned}$$

for any $(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in L$, $h \in \mathbb{R}$ and $k \in \mathbb{R}$. So every function $\psi(x_5, x_4, x_3, x_2, x_1)$ on G_5 extends uniquely as an invariant function $\widetilde{\psi}(x_5, x_4, x_3, x_2, t_2, x_1, t_1)$ on L

Theorem 2.1. *For every function $F \in C^\infty(L)$ invariant in sense (9) and for every $u \in \mathcal{U}$, we have*

$$u * F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) = u *_c F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \qquad (10)$$

for every $(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in L$, *where* $*$ *signifies the convolution product on* G_5 *with respect the variables* $(x_5, x_4, x_3, t_2, t_1)$ *and* $*_c$*signifies the commutative convolution product on* B *with respect the variables* $(x_5, x_4, x_3, x_2, x_1)$.

Proof: In fact we have

$$\begin{aligned}
& P_u F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \\
= \ & u * F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \\
= \ & \int_{G_5} F\left[(y_5, y_4, y_3, y_2, s)^{-1}(x_5, x_4, x_3, x_2, t_2, x_1, t_1)\right] \\
& u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & \int_{G_5} F\left[(\rho_1(s^{-1})(y_5, y_4, y_3, y_2)^{-1}, -s)(x_5, x_4, x_3, x_2, t_2, x_1, t_1)\right] \\
& u(y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & \int_{G_5} F\left[(\rho_1(s^{-1})((\rho_2(y_2^{-1})(-y_5, -y_4, -y_3)(x_5, x_4, x_3, x_2)), t_2 - y_2, x_1, t_1 - s)\right] \\
& u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & \int_{G_5} F\left[(\rho_1(s^{-1})((\rho(y_2^{-1})(x_5 - y_5, x_4 - y_4, x_3 - y_3), x_2, t_2 - y_2, x_1, t_1 - s)\right] \\
& u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & u *_c F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) = Q_u F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \qquad (11)
\end{aligned}$$

By the invariance of F, we get:

$$\begin{aligned}
& P_u F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \\
= \ & u * F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \\
= \ & \int_{G_4} F\left[(\rho(s^{-1})((\rho(y_2^{-1})(x_5 - y_5, x_4 - y_4, x_3 - y_3), x_2, t_2 - y_2, x_1, t_1 - s)\right] \\
& u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & \int_{G_4} F\left[x_5 - y_5, x_4 - y_4, x_3 - y_3, x_2 - y_2, t_2, x_1 - s, t_1)\right] \\
& u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\
= \ & u *_c F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) = Q_u F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \qquad (12)
\end{aligned}$$

where P_u and Q_u are the invariant differential operators on G_5 and B respectively.

2.2. Denote by $\mathcal{S}(G_5)$ the Schwartz space of G_5, which is the Schwartz space $\mathcal{S}(\mathbb{R}^5)$ of $\mathbb{R}^5$ let $\mathcal{S}'(G_5)$ be the space of all tempered distributions on G_5. If we consider the group G_5 is as a subgroup of L, then $\widetilde{f} \in \mathcal{S}(G_5)$ for $x_1$and x_2 are fixed, and if we consider B as a subgroup of L, then $\widetilde{f} \in \mathcal{S}(B)$ for $t_1$and t_2 fixed. This being so; denote by $\mathcal{S}_E(L)$ the space of all functions $\phi(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in C^\infty(L)$ such that $\phi(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in \mathcal{S}(G_5)$ for $x_1$and x_2 fixed, and $\phi(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in \mathcal{S}(B)$ for t_1 and t_2 fixed. We equip $\mathcal{S}_E(L)$ with the natural topology defined by the seminomas:

$$\phi \to \sup_{(x_5,x_4,x_3,x_2,x_1)\in B} |Q(x_5, x_4, x_3, x_2, t_2, x_1, t_1)P(D)\phi(x_5, x_4, x_3, x_2, t_2, x_1, t_1)| \qquad t_2, t_1 \ fixed \tag{13}$$

$$\phi \to \sup_{(x_5,x_4,x_3,t_2,t_1)\in K} |R(x_5, x_4, x_3, x_2, t_2, x_1, t_1)S(D)\phi(x_5, x_4, x_3, x_2, t_2, x_1, t_1)| \qquad x_2, x_1 \ fixed \tag{14}$$

where P, Q, R and S run over the family of all complex polynomials in 5 variables. Let $\mathcal{S}_E^I(L)$ be the subspace of all functions $F \in \mathcal{S}_E(L)$, which are invariant in sense (9), then we have the following result.

Theorem 2.2. *Let $u \in \mathcal{U}$ and Q_u be the invariant differential operator on the group B*, which is associated to u, then we have

(i) The mapping $f \mapsto \widetilde{f}$ is a topological isomorphism of $\mathcal{S}(G_5)$ onto $\mathcal{S}_E^I(L)$.

(ii) The mapping $F \mapsto Q_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, where Q_u acts on the variables $(x_5, x_4, x_3, x_2, x_1) \in B$.

Proof: (i) In fact $\sim$ is continuous and the restriction mapping $F \mapsto RF$ on G_5 is continuous from $\mathcal{S}_E^I(L)$ into $\mathcal{S}(G_5)$ that satisfies $R\circ \sim = Id_{\mathcal{S}(G_5)}$ and $\sim \circ R = Id_{\mathcal{S}_E^I(L)}$, where $Id_{\mathcal{S}(G_5)}$ $(resp.\ Id_{\mathcal{S}_E^I(L)})$ is the identity mapping of $\mathcal{S}(G_5)$ $(resp.\ \mathcal{S}_E^I(L))$ and G_5 is considered as a subgroup of L. To prove(ii) we refer to[25, $P.313-315$] and his famous result that is:

"*Any invariant differential operator on B, is a topological isomorphism of $S(B)$ onto its image*" From this result, we obtain that

$$Q_u : \mathcal{S}_E(L) \to \mathcal{S}_E(L) \tag{15}$$

is a topological isomorphism and its restriction on $\mathcal{S}_E^I(L)$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Hence the theorem is proved.

In the following we will prove that every invariant differential operator on $G_5 = ((\mathbb{R}^3 \times \{0\}) \rtimes_{\rho_2} \mathbb{R} \times \{0\}) \rtimes_{\rho_1} \mathbb{R}$ has a tempered fundamental solution. As in the introduction, we will consider the two invariant differential operators P_u and Q_u, the first on the group $G_5 = \mathbb{R}^3 \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}$, and the second on the group $B = \mathbb{R}^3 \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}$. Our main result is:

Theorem 2.3. *Every nonzero invariant differential operator P_u on G_5 associated to $\mathcal{U}$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image.*

Proof: In fact by equation (10), for every $u \in \mathcal{U}$ and $F \in \mathcal{S}_E^I(L)$

$$\begin{aligned} & P_u F(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \\ = & \int_{G_5} F\left[(\rho(s^{-1})((\rho(y_2^{-1})(x_5 - y_5, x_4 - y_4, x_3 - y_3), x_2, t_2 - y_2, x_1, t_1 - s)\right] \\ & u(y_5, y_4, y_3, y_2, s) dy_5 dy_4 dy_3 dy_2 ds \\ = & u *_c F(x_4, x_3, x_2, x_1, t) = Q_u F(x_4, x_3, x_2, x_1, t) \end{aligned} \tag{16}$$

for all $(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \in L$, where $\star$ is the convolution product on $\mathbb{R}^3 \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}$ and $*_c$ is the convolution product on the group $B = \mathbb{R}^3 \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}$. So the mapping $F \mapsto Q_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, then the mapping $F \mapsto P_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Since

$$R(P_u F)(x_5, x_4, x_3, x_2, t_2, x_1, t_1) = P_u(RF)(x_5, x_4, x_3, x_2, t_2, x_1, t_1) \tag{17}$$

then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_E^I(L) & \overset{P_u}{\longrightarrow} & P_u \mathcal{S}_E^I(L) \\ \sim\uparrow\downarrow R & & \downarrow R \\ \mathcal{S}(G_5) & \overset{P_u}{\longrightarrow} & P_u \mathcal{S}(G_5) \end{array}$$

Hence the mapping $F \mapsto P_u F$ is a topological isomorphism of $\mathcal{S}(G_5)$ onto its image.

Corollary 2.1. *Every nonzero invariant differential operator on G_5has a tempered fundamental solution.*

Proof **:** The transpose ${}^tP_u$ of P_u is a continuous mapping of $\mathcal{S}'(G_5)$ onto $\mathcal{S}'(G_5)$. This means that for every tempered distribution T on G_5 there is a tempered distribution E on G_5 such that

$$P_uE = T \tag{18}$$

Indeed the Dirac measure δ belongs to $\mathcal{S}'(G_5)$.

3 Hypoellipticity

3.1. As in [12]**,** the Lie algebra L_5, can be presented by the following matrix

$$L_5 = \begin{pmatrix} M_1(X_1, X_2, X_3, X_4) & 0_{4\times 4} \\ 0_{4\times 4} & M_2(X_1, X_2, X_3, X_4, X_5) \end{pmatrix} \mid X_i \in \mathbb{R} \tag{19}$$

where

$$M_1(X_1, X_2, X_3, X_4) = \left(\left[\begin{pmatrix} 0 & -X_1 & 0 & X_4 \\ 0 & 0 & -X_1 & X_3 \\ 0 & 0 & 0 & X_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid X_i \in \mathbb{R} \right] \right) \tag{20}$$

and

$$M_2(X_1, X_2, X_3, X_5) = \left(\left[\begin{pmatrix} 0 & X_2 & 0 & X_5 \\ 0 & 0 & X_2 & -X_3 \\ 0 & 0 & 0 & -X_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid X_i \in \mathbb{R} \right] \right) \tag{21}$$

Each X_i can be represented as the matrix with δ_{ij}. A matrix presentation of the group G_4 is thus the matrix exponential of L_5

$$\begin{aligned} G_5 &= Exp \begin{pmatrix} M_1(X_1, X_2, X_3, X_4) & 0_{4\times 4} \\ 0_{4\times 4} & M_2(X_1, X_2, X_3, X_5) \end{pmatrix} \mid X_i \in \mathbb{R} \\ &= \begin{pmatrix} N_1(x_1, x_2, x_3, x_4) & 0_{4\times 4} \\ 0_{4\times 4} & N_2(x_1, x_2, x_3, x_5) \end{pmatrix} \end{aligned} \tag{22}$$

where

$$N_1(x_1,x_2,x_3,x_4) = \left(\left[\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & -x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid X_i \in \mathbb{R}\right]\right) \quad (23)$$

and

$$N_2(x_1,x_2,x_3,x_4) = \left(\left[\begin{pmatrix} 1 & x_2 & \frac{x_2^2}{2} & x_5 - \frac{x_1 x_2^2}{2} \\ 0 & 1 & x_2 & -x_3 - xx \\ 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid X_i \in \mathbb{R}\right]\right) \quad (24)$$

3.2. It is easy to show, that is explicitly the basis of the Lie algebra is given by the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial_{x_1}},\ X_2(x) = \frac{\partial}{\partial_{x_2}} - x_1\frac{\partial}{\partial_{x_3}} + \frac{x_1^2}{2}\frac{\partial}{\partial_{x_4}} + x_1x_2\frac{\partial}{\partial_{x_5}} \\ X_3(x) &= \frac{\partial}{\partial_{x_3}} - x_1\frac{\partial}{\partial_{x_4}} - x_2\frac{\partial}{\partial_{x_5}}, X_4 = \frac{\partial}{\partial_{x_4}}, X_5 = \frac{\partial}{\partial_{x_5}} \end{aligned} \quad (25)$$

Any invariant differential operator on G_4 has the form

$$P = \sum_{\alpha,\beta,\gamma,\sigma,\theta} a_{\alpha,\beta,\gamma,\sigma,\theta} \left(X_1\right)^{\alpha} \left(X_2\right)^{\beta} \left(X_3\right)^{\gamma} \left(X_4\right)^{\sigma} \left(X_5\right)^{\theta}, a_{\alpha,\beta,\gamma,\sigma,\theta} \in \mathbb{C} \quad (26)$$

is solvable.

In particular the Laplacian operator on G_5

$$\Delta_{G_5} = (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 + (X_5)^3 \quad (27)$$

is solvable on the group G_5.Now, we consider on G_5 the following vector fields

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial_{x_1}} - (x_2+x_3)\frac{\partial}{\partial_{x_3}} - x_3\frac{\partial}{\partial_{x_4}} + x_1x_2\frac{\partial}{\partial_{x_4}} + \frac{x_2^2}{2}\frac{\partial}{\partial_{x_5}} \\ Y_2 &= \frac{\partial}{\partial_{x_2}} - x_1\frac{\partial}{\partial_{x_3}} + \frac{x_1^2}{2}\frac{\partial}{\partial_{x_4}} + x_1x_2\frac{\partial}{\partial_{x_5}} - x_3\frac{\partial}{\partial_{x_5}} \\ Y_3 &= \frac{\partial}{\partial_{x_3}} - x_1\frac{\partial}{\partial_{x_4}} - x_1\frac{\partial}{\partial_{x_3}} - x_2\frac{\partial}{\partial_{x_5}} \\ Y_4 &= \frac{\partial}{\partial_{x_4}}, X_5 = \frac{\partial}{\partial_{x_5}} \end{aligned} \quad (28)$$

Our main results are

Theorem 3.1. *The operator*

$$Y^2 = \sum_{i=1}^{5} Y_i^2 \tag{29}$$

is solvable and hypoelliptic on G_5

Proof: Define the mapping $\Gamma : \mathcal{D}'(G_5) \to \mathcal{D}'(G_5)$ as follows

$$\Gamma\phi(x_5, x_4, x_3, x_2, x_1) = \phi(x_5+\frac{1}{2}x_1x_2^2-x_2x_3, x_4+\frac{1}{2}x_1^2x_2-x_1x_3, x_3-x_1x_2-x_1x_3, x_2, x_1) \tag{30}$$

The operator Γ is hypoelliptic and has an inverse, which is

$$\Gamma^{-1}\phi(x_5, x_4, x_3, x_2, x_1) = \phi(x_5-\frac{1}{2}x_1x_2^2+x_2, x_4-\frac{1}{2}x_1^2+x_1, x_3+x_1, x_2, x_1) \tag{31}$$

It is easy to show that the operator Γ verifies the following equation

$$\Gamma^{-1}(\partial_{x_5x_5} + \partial_{x_4x_4} + \partial_{x_3x_3} + \partial_{x_2x_2} + \partial_{x_1x_1})\Gamma = \sum_{i=1}^{5} Y_i^2 \tag{32}$$

where $\partial_{x_5x_5} + \partial_{x_4x_4} + \partial_{x_3x_3} + \partial_{x_2x_2} + \partial_{x_1x_1}$ is the Laplace operator on $\mathbb{R}^5$. So the solvability and hypoellipticity of the operator $\sum_{i=1}^{5} Y_i^2$

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